Notes on notes on the plane

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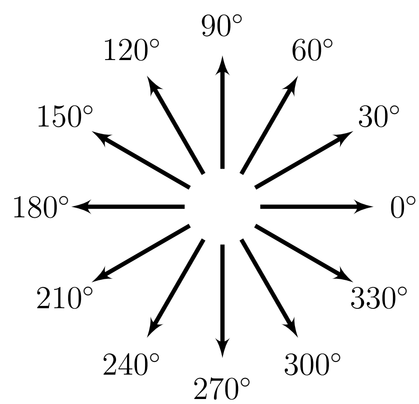
1 Notes on the plane

Suppose you are an ant or other point-like creature living on the two-dimensional Euclidean plane. You can move in any direction; for instance, the twelve multiples of 30° shown here.



Those twelve directions have a circular structure: 0° is one step of 30° counterclockwise of 330° and 330° is one step clockwise of 0°. You can turn as many degrees clockwise or counterclockwise as you want, and only get dizzy; you'll never run out of directions.

It happens that the musical octave has a similar cyclic structure. Assume octave equivalence and 12 equally spaced notes in the octave, hereinafter

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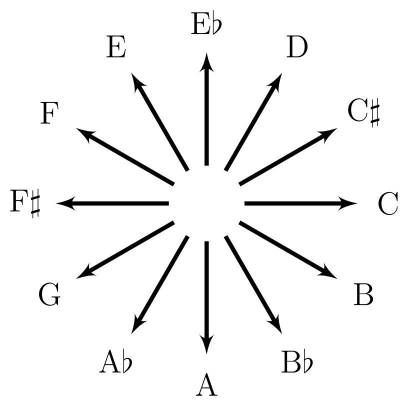
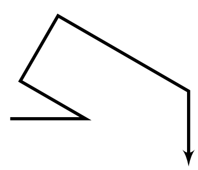
called 12-EDO ("twelve-part equal division of the octave"). Both of these are traditional assumptions in the music of Western civilization. Then you can draw a diagram of musical notes that looks almost exactly like the diagram of directions on the plane.



Take a short walk.



Your path defines a sequence of angles and distances.

0° for one time unit
150° for one time unit
60° for one time unit
330° for two time units
270° for one time unit

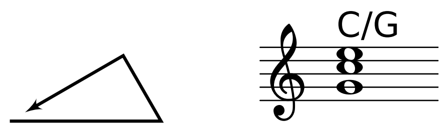
Replace those angles with the corresponding notes, and it's music.

A walk (consisting of a sequence of points on the plane and the line segments connecting them) can be mapped naturally into a sequence of notes, and vice versa.

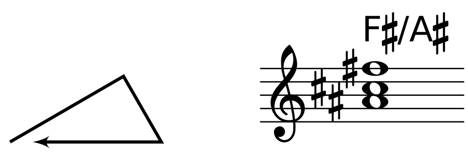
2 Chords

For the purposes of this section, suppose there is no such thing as time.

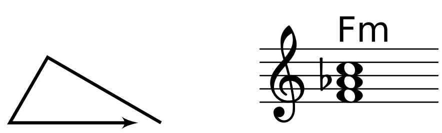
Consider a 30°-60°-90° triangle. Walk around it, play the resulting three notes simultaneously, and it's a chord. Note the geometry doesn't specify any particular voicing. I chose this one for the picture just to make the notes look nice on the staff.



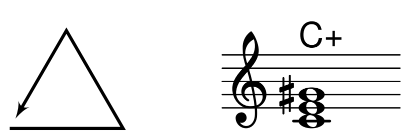
Walking around the same triangle in the opposite direction gives F♯ major, which is the tritone substitution of C major. Note a tritone (six units of 12-EDO) corresponds to 180° on the plane. Each of the directions has been replaced by its opposite, and each of the notes by its tritone partner. By rotating this triangle to other orientations, we can derive any of the 12 major chords.

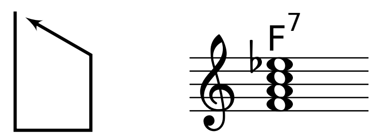


The mirror image of this triangle generates a minor chord.



Other triangles, and polygons with more than three sides, generate other kinds of chords.

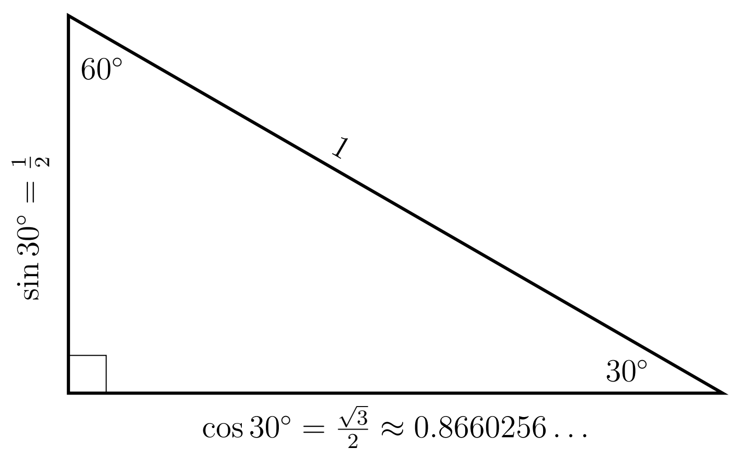




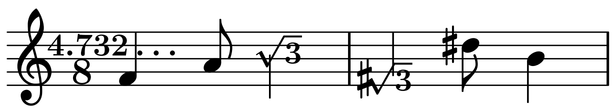
We can learn things about the chords by looking at the polygons. For instance, a chord contains a tritone if and only if its polygon has two parallel sides. Or maybe not. That would be true of a convex polygon, and for every chord there exists at least one convex polygon; but the convex polygon is not necessarily unique, and nonconvex polygons also exist for some chords, including nonconvex polygons with parallel pairs of sides for chords with no tritones. See if you can find examples.

3 Microtones and microrhythms

A triangle with convenient angles that are multiples of 30° will usually have side lengths in irrational proportion to each other.

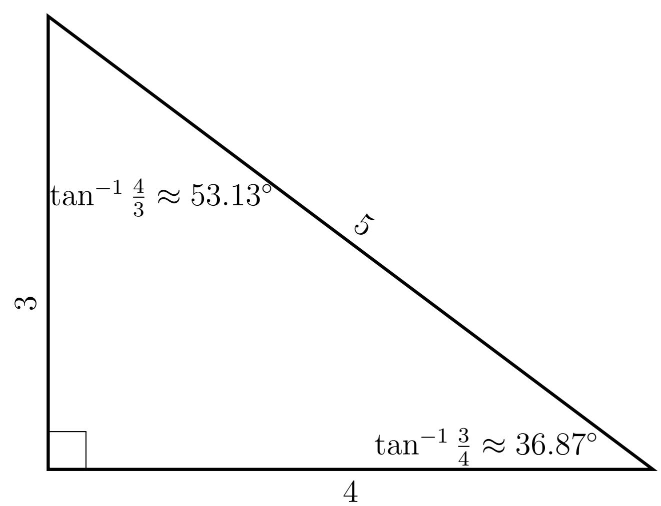


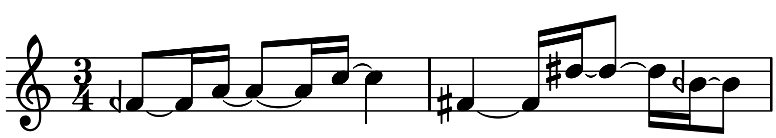
Let's remember that time exists, and switch from the chords of the previous section back to the idea of making melodies. We will travel among the vertices of the triangle, using the side length as note durations and the direction of movement to choose pitches within the octave, using some other means not yet specified to choose the octaves themselves. We are going to have trouble with the rhythm.



There is a thing in music theory called an irrational time signature. That is a time signature in which the denominator is not a power of two. The above time signature, with an irrational number in it divided by $8 = 2^3$, is therefore rational. Note that it also isn't as simple as a polyrhythm with voices playing at an irrational tempo ratio to each other [5]; here there are notes with irrational duration ratios in the same voice.

Maybe we could try to make the triangle side lengths integers.





In this case the rhythm comes out all right, but we have some strange notes that aren't quite on the 12-EDO scale; and the 3-4-5 triangle is a relatively tame example because of its embedded right angle, which gives a perfect 300¢ 12-EDO minor third. The roughly 6.87° deviation from a 30° boundary translates to about 22.9¢ of pitch, very close to one ninth of a 200¢ whole tone. That nicely coincides with one of the special microtonal accidentals used in Turkish classical music [13]; but we also need the standard sharp sign used in its standard (50¢) sense instead of with its overloaded Turkish meaning.

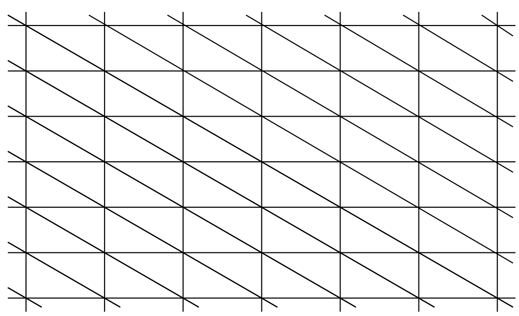
Maybe we could fix things by using some other way of measuring distance instead of the Euclidean L_2 metric. In particular, we might use the L_1 metric, which guarantees rational distances if all the points are at rational coordinates. That is a subject for future research; in the current note, I assume Euclidean distances.

Things will get even more complicated if we try to rotate the triangle to other positions, and most integer-side triangles are not right triangles, and have *all* their angles irrational. So if we are not extremely careful with the geometry, we will get strange tuning, strange rhythms, or most likely both, and the resulting music will sound *fucked up*.

However, music theorists prefer to describe this kind of thing with the term *avant garde* [17], and they can be convinced that it's good, if you contextualize it carefully.

4 Be vewwy quiet

Suppose the entire Euclidean plane is tiled with 30°-60°-90° triangles.



A hunter, "Elmer," traverses the edges of this tiling at a constant speed. When he arrives at a vertex, he chooses one of the edges from that vertex through which to leave, by some means yet to be defined. He also starts singing [1] a note, at a pitch determined by his new direction. The duration of the note will be determined by the length of the edge Elmer is traversing, because his speed is constant. When he reaches the next vertex, he ends the note and starts another.

Elmer is chasing a waterfowl, "Daffy." Daffy moves according to a *Cauchy flight*. That is a random walk with a Cauchy heavy-tailed distribution of step sizes. If a strobe light flashes, catching Daffy's current location, then after a small time increment h, when it flashes again his new location will have shifted by a vector whose direction is chosen uniformly at random and whose length is the ratio of two standard normal random variables, perhaps scaled by some appropriate amount. Ideally, we want the scaling to work so that this will be stable as we take the limit of h going to zero.

This kind of random walk is somewhat like the familiar Brownian motion based on the Gaussian distribution, but in the Cauchy flight, Daffy will tend to hang around in a small neighbourhood for a while, and then make a sudden leap for a long distance to a new neighbourhood—more so than would be the case with standard Brownian motion. Figure 1 illustrates it. Such heavy-tailed random walks are of interest to the people who study things like the stock market, in which sudden large surprising events (called *black swans*) tend to occur more often than would be predicted by the light-tailed Gaussian distribution [23, 24]. Because of his flight path, we classify Daffy not as a duck but an Australian black swan, *Cygnus atratus*.

The set of vertices and edges of the tiling, to which Elmer is confined, has measure zero. But the support of the random variable of Daffy's position is the entire plane. The probability that Daffy occupies a vertex or point on an edge of the tiling is zero. Thus, even if he might be able to run faster than Daffy's average speed,[1] *Elmer can never actually catch Daffy*. He can only chase him forever.

Elmer is nonetheless vewwy optimistic, and will continue indefinitely to attempt to come as close as possible to catching the black swan. That influences his rule for choosing edges: at each vertex, where he is faced with a choice of new direction, Elmer will turn to leave along whichever edge minimizes the angle to the vector from his current location to Daffy's (that is, $D - E$). However, being superstitious about the *diabolus in musica*, Elmer will never perform an instantaneous 180° turn and double back on the edge along which he arrived (singing a tritone in the process). If that would be the nearest edge to the vector $D - E$, then he will choose the second-nearest edge instead. If Daffy did not move, he would with probability 1 nest in the interior of some triangle, and Elmer would orbit him forever, singing a

[1] Daffy does not actually have such a thing as an "average" speed, due to the spectacularly loony statistics of the Cauchy distribution, analysis of which includes steps like "okay, now let's differentiate this function one and a half times. . . "; but Elmer would still be in trouble even if Daffy were following a better-behaved type of random walk with a more normal (for instance, a normal) distribution of step lengths.

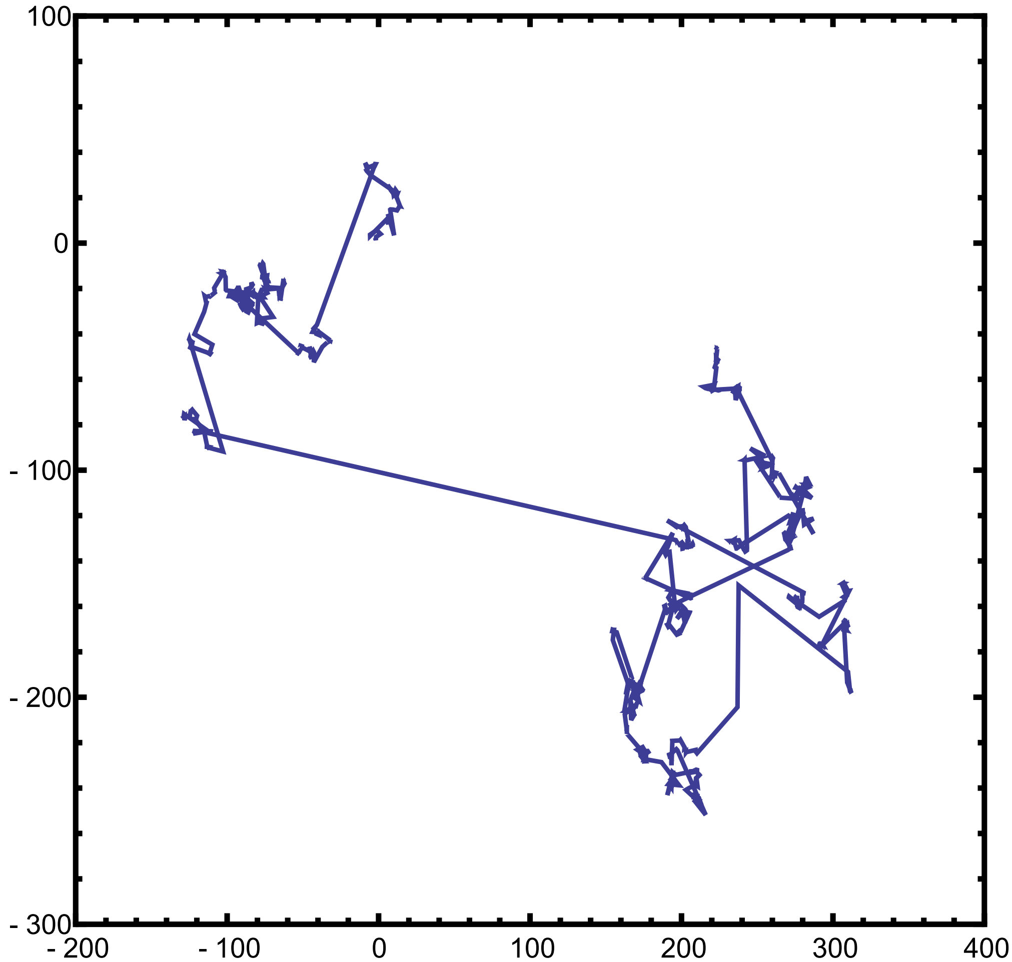
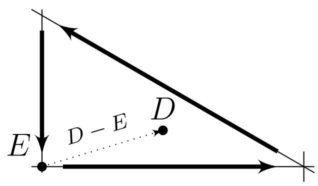
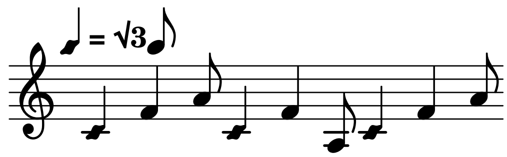


Figure 1: Cauchy flight (a special case of the Levy flight). Illustration from Wikipedia user “PAR,” public domain. [16]

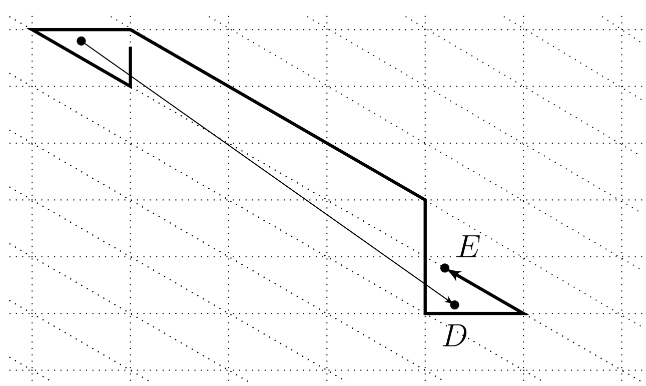
three-note arpeggio of a major or minor chord (here F major).

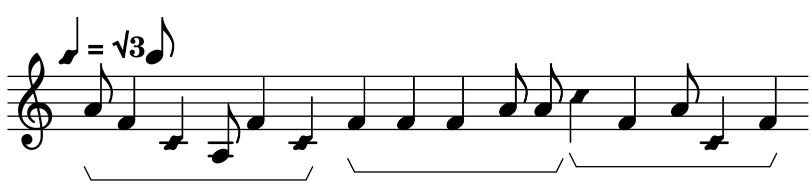




Note the special note heads for radiculotreblequavers. I have given up trying to write a time signature for this melody; even if I could make it work with just Elmer, it would break when I introduce Fanny, Grover, and Hermione, later. But before meeting them, let's make an observation: the tonal context of Elmer's tune, meaning the notes he can choose from and the kind of sequence in which he is likely to choose them, is set by the tiling we imposed on the plane. If we rotated the tiling, Elmer would sing in a different key. If we changed the shape of the triangles, Elmer would arpeggiate some other kind of chord instead of a major triad. The tiling defines many but not all aspects of the music Elmer will sing.

Meanwhile, Daffy's location defines other aspects of Elmer's tune. Consider how the music changes if Daffy makes a black-swan leap to a relatively distant location.





In the first segment, with Daffy in his original location, Elmer circles him, forming an F major arpeggio (A F C A F C). Then, Daffy makes his move.

Elmer spends the second segment seeking the new location (F F F A A). That is rhythmically and melodically much different from the first segment: repeating notes of the same duration instead of a pattern of changing pitches and durations. Finally, Elmer reaches the triangle that contains Daffy's new location, and he settles back into the arpeggio pattern (C F A C F A); but the pattern has changed. It's the same F major chord, but arpeggiated upward instead of downward, and the durations of the C and A have been exchanged.

We can observe something important here: the structure of Elmer's tune echoes the structure of Daffy's flight. Daffy's Cauchy flight is characterized by periods of relative immobility separated by black swan jumps; and Elmer's tune is respectively characterized by periods of repetitive focus on a single chord, and less structured connecting episodes in which a repeated pitch and duration drives the music forward.

If Daffy made a longer jump, then it would take longer for Elmer to catch up to him, and during that time, Daffy might also make a second jump. So Elmer might have to change direction during his chasing phase, resulting in a more complicated structure. In fact, the Cauchy flight has a *fractal* structure [14], having the same general appearance at multiple scales. Up close (over short durations), Daffy sits more or less still for a while, then makes a jump. Several of those nesting points joined by relatively short jumps form a cluster; but clusters themselves are joined by rarer, longer black swan jumps. Over a longer time period, Daffy's locations will form a cluster of clusters; but then eventually, inevitably by his nature, he will make an especially large black swan jump, out of the supercluster and into a new one. (Refer again to Figure 1.) The multi-scale self-similar structure of the flight of the black swan will be reflected in Elmer's path (and therefore melody) as he chases Daffy.

This idea of structure at multiple scales is widely believed to be a common characteristic of beautiful things both natural and artificial, and of music in particular [4, 15].

5 That left turn at Albuquerque

Elmer's tune as described will have a multi-scale fractal structure, desirable for convincing serious music theorists that what we are doing is worthwhile, but it will still sound quite boring, because there are only really five tonal and melodic things that can occur: Elmer can sing a B major or F major triad as

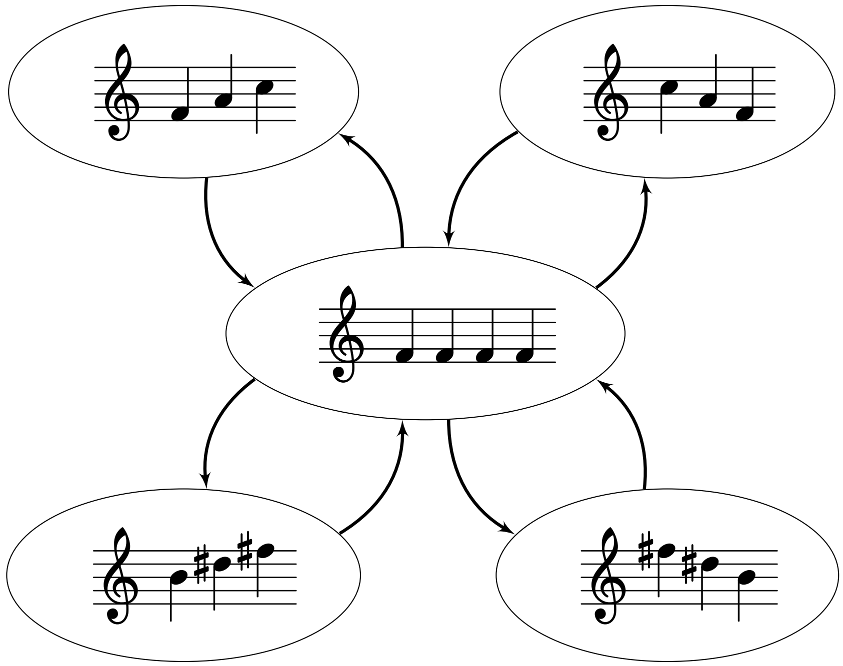
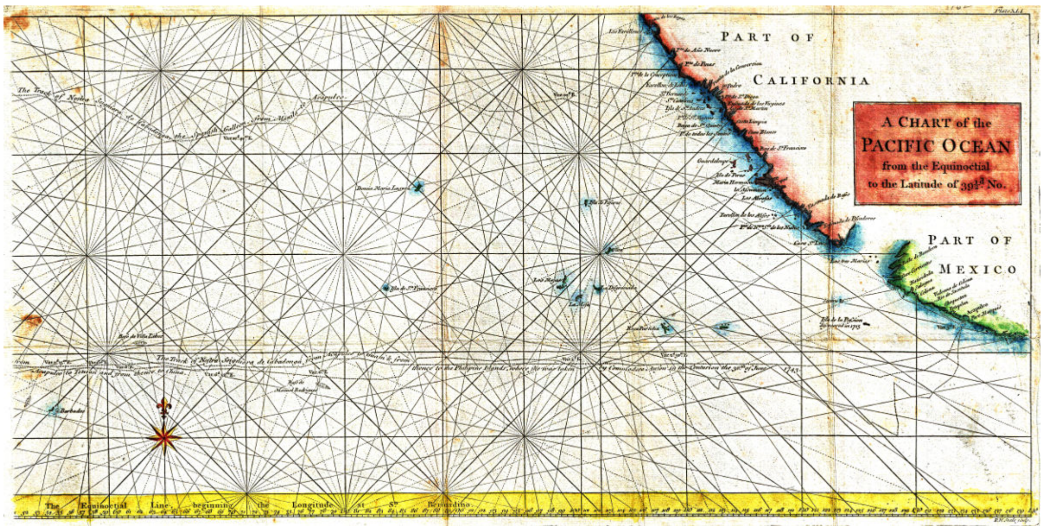


Figure 2: Worst song, played on ugliest guitar

an arpeggio, each either upward or downward (a total of four possibilities), or he can repeat one or two notes for a while as he runs to catch up with Daffy, at which point he will switch back to one of the four arpeggios, selected basically at random. Through all this, he only ever gets to sing six pitch classes. We can describe the entire tonal and melodic content of Elmer's tune by the finite state machine of Figure 2.

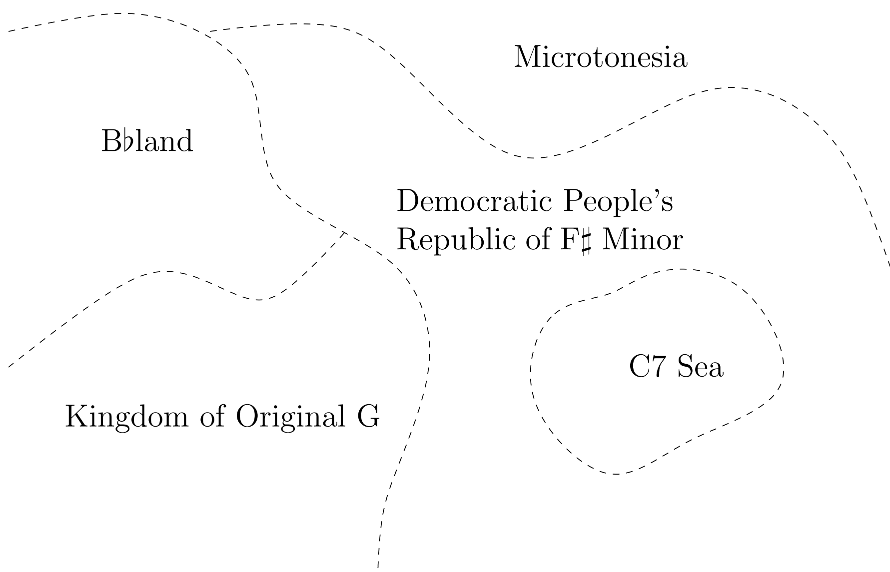
This limitation on harmony and melody comes from the boring tiling of the plane that we adopted back at the start of Section 4. The entire infinite plane is covered with just one kind of triangular tile (the 30°-60°-90° triangle, which signifies a major or minor triad), and the tiles occur in only two orientations and one handedness (corresponding to B major and F major). They also occur in only one size (fixing the rhythm of the music to contain only three note lengths—and moreover each pitch class only ever occurs in one of these note lengths). Although Daffy's fractal random walk includes interestingly-distributed black swan leaps to new contexts corresponding to new locations on the plane, he might as well not bother, because they all sound the same anyway.

Suppose instead we used a more interesting collection of lines and points on the plane [3].



A small section of the plane will contain lines at only a few slopes, corresponding to a few pitch classes that define a scale and key. The number of distinct lengths of line segment in a small area will be similarly limited, defining a rhythmic context. The local graph structure of the vertices and edges defines which notes have which lengths and the sequence in which they follow each other. All these factors taken together express something much like a *raga*, as in Indian classical music; and this *raga* is connected to one location on the plane, with a slightly or completely different one defined by any other location.

As long as Daffy and therefore Elmer remain in a small area of the map, they will generate music with a distinct feeling and texture, the *raga* of that geographic location. But when Daffy's flight leads them to a different geographic location, that feeling will change. We can imagine a map designed, or ideally, automatically generated, to have different realms generating a diversity of musical textures and with different keys emphasized. The multi-scale flight of the black swan would then visit those regions in a sequence that would create large- and small-scale musical structure in a natural and beautiful way.

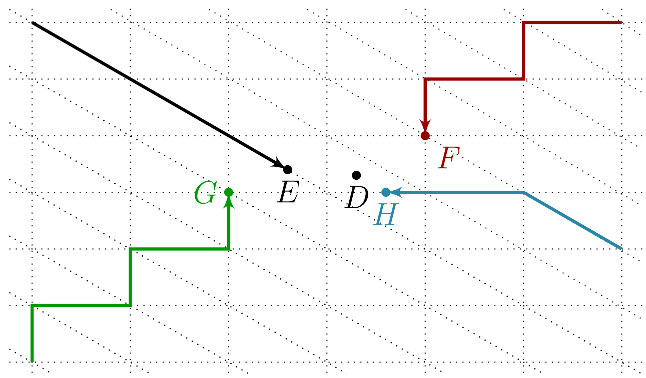


Ideally, the map should not just be distinct regions pasted together, but (like real geography) characterized by continuous change with no homogeneous regions, even if there are also boundaries and linear and point features. Two places close together should be similar, and two places further apart should be more different. The changes cannot be limited to a few liminal spaces (that is, boundaries), either. Rather the whole plane should be liminal, with interesting shifts in texture apparent near every location, regardless of the scale we use to define "near." That is a description of fractal geometry. The geometric graph defining Elmer's note pitches and durations should be fractal like Daffy's black swan flight.

How can we construct such a set of vertices and edges?

6 Hunting in packs

Suppose there are several more hunters active simultaneously; not only Elmer, but also Fanny, Grover, and Hermione. If they are far apart on the plane, they will all follow different paths and create different melodies. Letting them all sing at once creates polyphonic music.



If the hunters all follow the same movement rules, independently of each other, then it is much easier for them to start making the same decisions than to stop doing so. Very quickly we will end up with all four hunters orbiting Daffy on the same triangle; and even after Daffy jumps to another part of the plane, the hunters will all choose identical or nearly identical paths to chase him. The polyphony will be boring, even if we succeed in the efforts of the previous section to create an interesting landscape which makes individual parts interesting.

One thing we can do is give each hunter a different running speed. If Hermione is faster than Grover who is faster than Fanny who is faster than Elmer, then when Daffy makes a large jump, Hermione will be the first to catch up to him, and will start singing arpeggios while the others are still in the repetitive-note chase phase. Then Grover will join her, then Fanny, but before Elmer arrives, oh! Daffy may take another flying leap. A hunter's speed creates a kind of filter on the scales at which he or she reacts to the movement of the black swan. Faster hunters respond more to the small-scale movements, while slower hunters react only to the large-scale, less frequent, black swan leaps. This effect can keep the voices from boringly duplicating each other, not only in terms of pitch but also in the nature of their musical structure.

However, different low-pass filters of the same fractal flight can only differ so much, and we may hope for better. We also might hope to get more interesting rhythms and harmonic structures into each individual part; bottoming out in three-note arpeggios whenever a hunter gets near the black swan seems like a waste of all this effort.

Let's add another rule to the hunters' behaviour. When a hunter arrives at a vertex and needs to choose an edge through which to leave, we already said they would choose the one with the smallest angle to the direction of Daffy, excluding the case of turning 180° to return the way they came. Let's say that each hunter will also avoid ever choosing *the same note currently in use by some other hunter*, resolving the case of two hunters arriving at vertices simultaneously in some arbitrary way. Thus if Elmer arrives at a vertex, and the earlier rules would have him leave by an edge at 90° and sing D♯, but Fanny is currently singing D♯, then Elmer will choose some other edge.

With k hunters, a hunter may be forbidden from choosing up to k edges out of each vertex (one for each of the other hunters, and the one along which they arrived themselves). If there are any vertices of degree less than or equal to k, which seems like it might easily occur, then there may be a problem of no available edges remaining. In that case, we will let the hunter break the no-duplication rule and just choose the edge nearest the direction to Daffy (but not their own arrival edge), even though it duplicates another hunter's note; just as in the case of Elmer operating alone.

This rule change has two important consequences. It (directly) causes hunters to choose distinct pitch classes, so all the hunters together will tend to form chords; and it (indirectly) influences the melodies produced by individual hunters. Elmer is no longer likely to just orbit Daffy in a three-note sequence, because quite often the next note in the sequence will happen to be grabbed by some other hunter first. Each time that happens, he must choose some other edge, which in turn moves him further away from Daffy and may disturb his next few moves as well. There will still be a tendency to orbit, but the orbits will be much less repetitive.

Since all the hunters will tend to be attracted by the black swan to the same region of the plane, their pitch and rhythm choices will all tend to be similar to each other at any given point, reflecting the angles and edge lengths prevalent in that part of the plane. The map upon which the hunters move ought to be designed to contain angles and edge lengths that will lead to harmonious music, at all locations but in a different way in each location.

At this point it is natural to make a Big Wild Guess.

Big Wild Guess

Visibly interesting tilings of the plane will produce audibly interesting music.

7 The Black Swan Suite

I have now introduced most of the theory behind my musical work *Black Swan Suite* [22], which you can hear at `http://audio.northcoastsynthesis.com/index.php?a=blackswan`. I suggest listening to it before, or while, you read the following commentary.

I wrote software in C to generate tilings of the plane, a different one for each of the five movements, and compute the note sequences basically according to the rules described above. The black swan's motion is (pseudo)random, but the hunters' is roughly deterministic. I did a certain amount of manual adjustment of the times and locations to add and remove hunters, their speeds and voices, and so on, in an effort to get the music to sound good. A certain amount of the musical structure of each movement comes from those manually-imposed decisions; but very much of it emerges naturally from the fractal structure of the black swan's flight path.

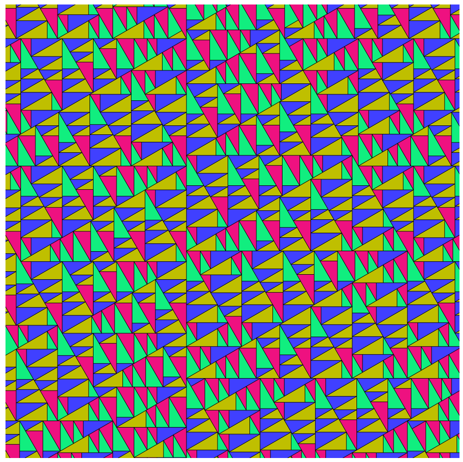
There are a number of relatively uninteresting technical issues I do not propose to go into in detail here, such as the exact structure of the Csound [25] software synthesis instruments I used, and handling of certain boundary cases in the movement rules for the swan and hunters. In some movements I used inharmonic timbres to try to make the intervals sound more consonant [19]. If there is a lot of attention given to this work I might write some more about those issues.

I have up to this point glossed over the question of how to choose the octave for a note. In general (with some local modifications in some movements of the piece) the rule is that a hunter will choose the octave that places the new pitch as close as possible to the previous pitch in their voice. That may cause the pitch to run away upward or downward if the hunter is orbiting a face of the embedding; so each hunter also has a configurable upper and lower pitch limit and will skip up or down an octave where necessary to keep pitches within their range.

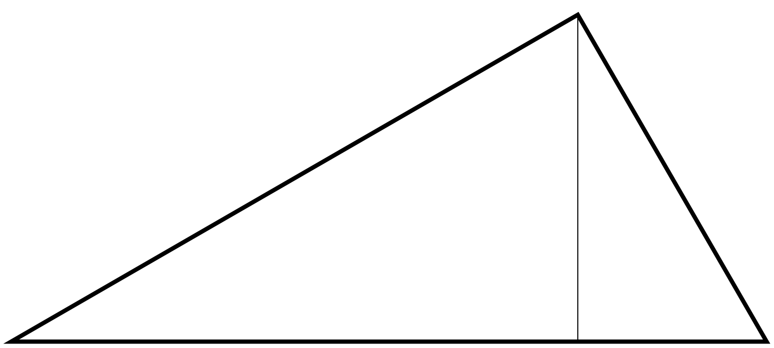
In order to make the rhythms more interesting, especially in some of the unit-edge tilings, hunters also have a configurable setting for tying or slurring notes. When this flag is turned on and two consecutive notes are less than a tritone apart, they will be slurred or tied, through as many successive notes as this criterion continues to be met.

The only big question remaining is, what kind of tiling will we impose on the plane?

7.1 The Nest



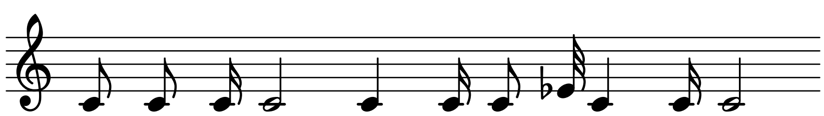
The tiling for this movement is very simple, being a sort of warm-up for the more complicated things done in later movements. Start with a 30°-60°-90°. Draw an additional segment from the right-angle vertex, so that it meets the hypotenuse at another right angle. This trisects the right angle, but trisecting right angles is harmless enough.



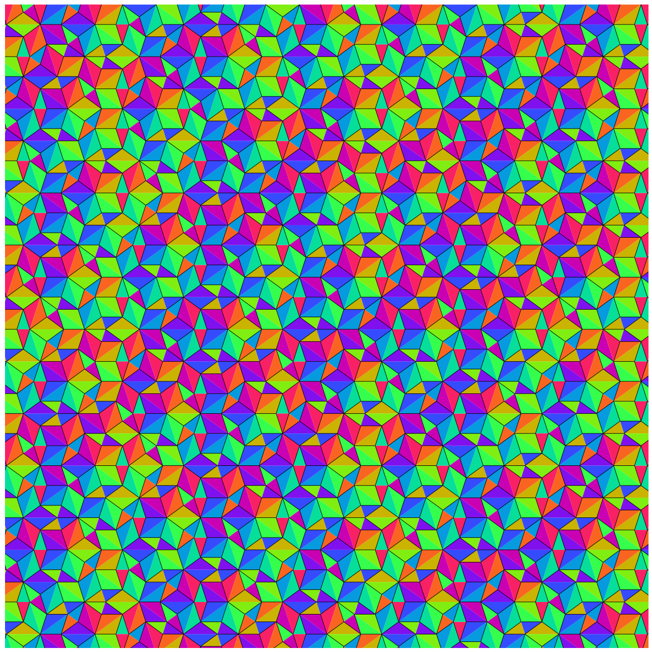
We have divided the 30°-60°-90° triangle into two smaller 30°-60°-90° triangles, one of them having the opposite handedness to the original. Choose a minimum acceptable size for triangles, and repeatedly apply this division process to any triangles larger than the minimum size until none remain.

This tiling procedure generates triangles that are all the same shape, occur in four different orientations, and have many different sizes, as can be seen in the image at the start of this subsection. The triangles in that image are given colours determined by their orientation; with each corresponding to a tritone-partner pair of two major or two minor chords, this tiling produces a basic vocabulary of eight chords. It also includes line segments at four different slopes, which (traversed in two directions each) gives a scale of eight notes. All eight are included in the standard 12-EDO vocabulary. The range of different sizes gives a range of note lengths, erasing almost all sense of regular beat or rhythm. Since the tiling looks substantially the same everywhere, there is very little feeling of tonal or rhythmic movement as the black swan moves to different locations. The overall feeling of the music is simple and static.

It can also be seen that the edges tend not to meet in "nice" intersections; often, the edge of a higher-level triangle will form a dividing line, such that edges meeting it on one side do not line up with edges meeting it from the other side. Very few sequences of consecutive segments of the same line persist for a long distance across the space. This typical geometric feature of the tiling creates an audible feature in the music during chase phases: sequences of notes on the same pitch with varying duration, interrupted by single brief notes at other pitches as the hunter jogs to one side or another to switch to a parallel line. Listen for this feature.



7.2 The Pavement



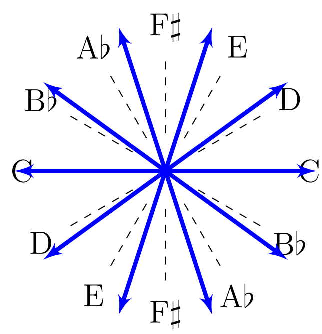
Penrose tilings [11, 6] require very little introduction. These are some of the earliest and most famous of the nonperiodic tilings; they opened up the field. In this movement of the suite, I'm using a Penrose rhombus tiling generated from Robinson triangles [8], largely because I can create that by triangle-subdivision rules much like those used in the previous movement. The subdivision rules actually generate a tiling with two kinds of triangles (visible from the colours in the picture); my code marks and ignores edges in order to merge pairs of triangles into the rhombuses that make up the Penrose tiling.

This tiling has a kind of multi-scale structure, something like the fractal structure originally desired: any small patch of the tiling will be repeated an infinite number of times in different parts of the plane, but there is no periodic structure describing the entire plane as simple repetition of a finite-sized pattern. I'm not sure to what extent that property may be audible in the music.

All the edges are the same length, leading to a fixed note duration for each voice (possibly different lengths for different voices, because the hunters move at different speeds). I used the slur flag a fair bit to keep the rhythm interesting.

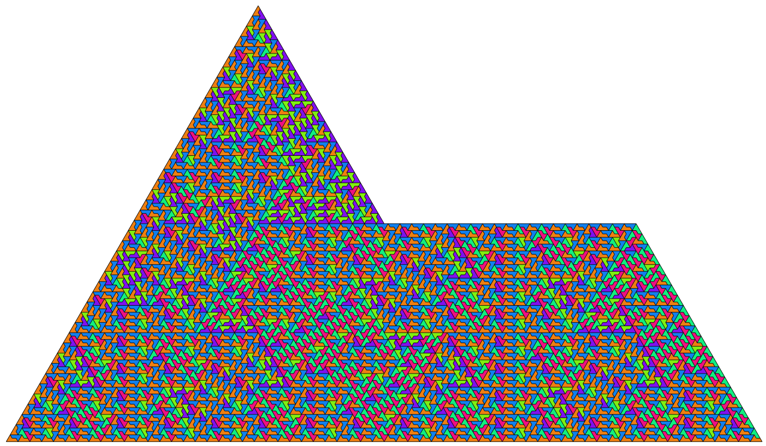
This Penrose tiling contains edges at five different slopes, giving notes on a 10-EDO scale [2]. That shapes its harmonic structure a great deal: many of the familiar intervals of 12-EDO music are absent, and other, alien, intervals occur. Furthermore, because the tiles are rhombuses—that is, parallelograms—each face of the embedding consists of two pairs of parallel edges. A hunter orbiting such a face will sing very many tritones, as well as the strange 10-EDO intervals. The result may not sound very musical. There is nothing for the ear to grab hold of.

I added another configurable setting to the hunters to allow them to double their angles. Instead of the original mapping of the 360° compass rose to the notes of the 12-EDO scale, the angle-doubled mapping looks something like this. Rotation by 180°, for instance by traversing an edge in the opposite direction, does not change the pitch.

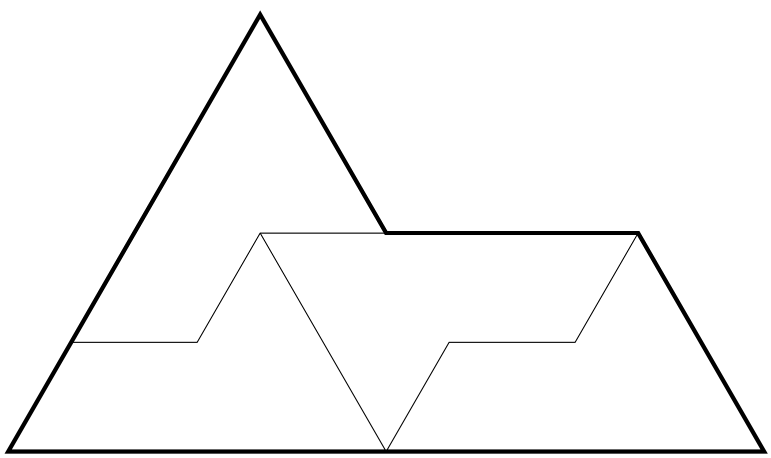


Note that the hunters do not, in general, actually sing the named 12-EDO (well, in fact 6-EDO) notes shown on the diagram. Those notes are only to illustrate the mapping of *two* octaves onto the circle. The hunters who have the angle-doubling flag set actually sing the unnamed notes of the 5-EDO (five notes per octave) scale indicated by the bold arrows. The 5-EDO tuning is sometimes, though controversially, claimed to be part of the Indonesian *gamelan* tradition [20, Chapter 10]. My hope is that it will sound at least a little more musical and human than the more purely mathematical 10-EDO; by using it earlier in the piece and then removing the flag and allowing the hunters to sing all ten tones of 10-EDO, my intention is to ease the listener more gently into the alien tonal realm of the Penrose tiling.

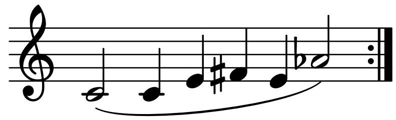
7.3 The Dunes



This tiling is based on the *Sphinx hexiamond* [9, 12]. Join two equilateral triangles along a shared edge and you get a *diamond*. Join several to form a shape and you have, in general, a *polyiamond*; a *hexiamond* is a polyiamond made from six equilateral triangles. The Sphinx hexiamond in particular is special because it can be divided into (or assembled from) four smaller copies of itself; not many pentagons can be divided in such a way.

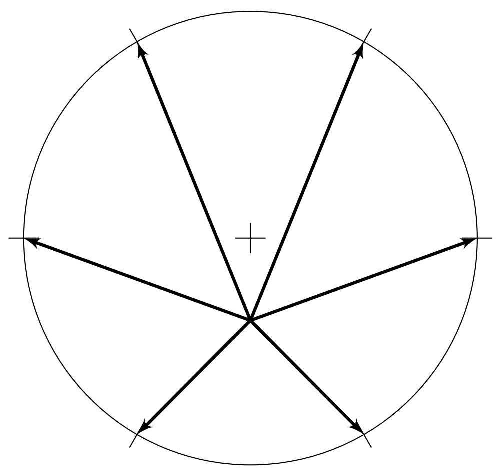


Repeating the dissection recursively generates the tiling for this movement of *Black Swan Suite*. There are only three edge lengths in the result, in a strict 1:2:3 proportion, so this movement ends up with a very strong rhythmic character. Because all the tiles are the same shape (albeit different orientation and handedness), there is a strong tendency to repeat the basic figure corresponding to a circuit around the tile.

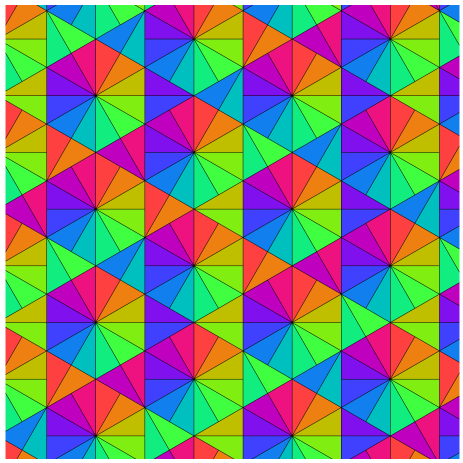


Lines occur at three distinct slopes, so the basic tuning would naturally be 6-EDO: the "whole tone scale" of the music of Western civilization. I thought that was a little boring, so in an attempt to give the movement a bit more of a tonal focus, I retuned the six notes of the 6-EDO scale.

First, I mapped the six notes into equally spaced points on a circle, forming 60° intervals around the centre. Then, I chose an off-centre point, and found the angles from that point to the six notes. The resulting angles, not equally spaced, were used to generate the pitches for the six notes of the scale used in this movement.



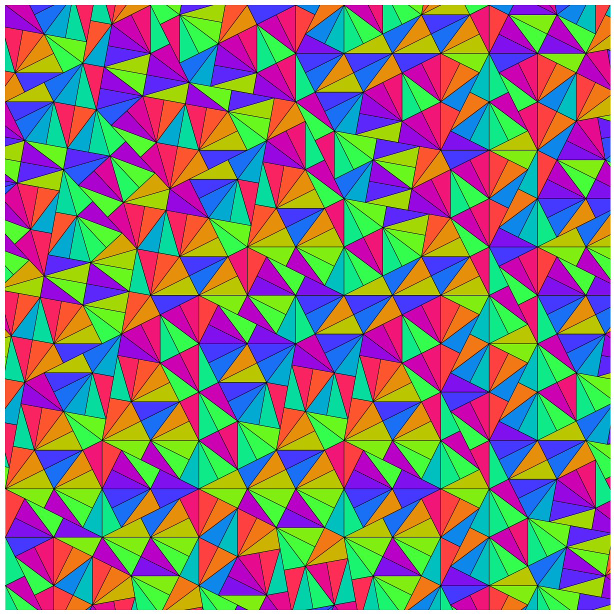
7.4 The Capture



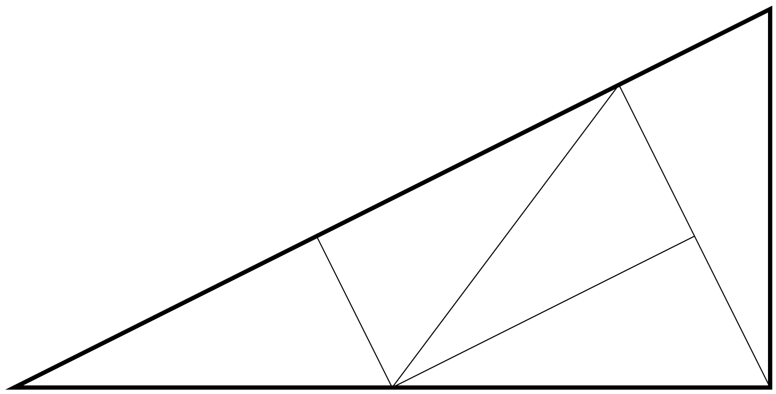
This trihex tiling [10] looks quite repetitive, and it is. The nonperiodicity appears in the little triangular areas in between the hexagons. In some sense it is just barely nonperiodic: every patch contains at most one tile not contained in some periodic subset of the tiling. From a musical perspective, the degree-12 vertices at the centres of the hexagons tend to create chromatic melodies, where any note can follow any other.

Since the triangles are 30°-60°-90°, this is another movement in 12-EDO; but unlike the first movement, all 12 notes really are used.

7.5 The Escape



The Conway-Radin pinwheel tiling [7, 18] is based on a right triangle that is one half of a *domino* or 2×1 rectangle, sliced diagonally. That makes the non-right angles roughly $26.56°$ and $63.43°$, and the subdivision rule generates copies of the original triangle tilted at those angles to each other. Since the *angles*, not just the side lengths, are irrational, the subdivision carried to its limit generates lines at an infinite number of different slopes. Whereas the tilings used in other movements generated music in 12-EDO, 6-EDO, and 10-EDO, this tiling in effect generates ∞-EDO: an unlimited number of different microtonal notes. The underlying triangle, nonetheless, is close enough to $30°$-$60°$-$90°$ to create something recognizable as a major or minor chord when a hunter orbits it.



Since the five triangles generated by the subdivision rule also include a pair of two in identical orientation, and a pair related by a 180° rotation, there is also a strong tendency for widely-separated sections of the tiling to duplicate each other, within a limited range. That gives this tiling the kind of self-similarity also found in many other nonperiodic tilings. It means that after a black swan leap, the swan and hunters may end up in a different, but familiar, neighbourhood of the tiling; imperfect long-range correlations in space give imperfect long-range correlations in time, as desired for music.

At the same time, the infinite number of line slopes means there is an endless variety of keys, and of more general tonal contexts, for the agents to explore. Music generated from this tiling will repeat keys, but choose them from an infinite set. Finally, because all the triangles in the substitution rule are the same size, the number of distinct edge lengths remains small if the substitution is applied a uniform number of times across the entire plane. The edges are irrationally related ($1{:}2{:}\sqrt{5}$), so no fixed time signature will work well, but there is still a perceptible rhythm. We do not see the arbitrarily small edges that arose in the 30°-60°-90° subdivision of the first movement. All the coordinates of vertices are rational numbers, which could lead to interesting rhythmic consequences if we used an alternate metric for the side lengths. In this piece, however, I used the Euclidean metric.

A finite piece of music obviously[2] cannot actually use an infinite number of notes, even if written on a scale that defines them. The subdivision is only carried out a finite number of steps in practice, and the recording only lasts a finite time. In fact, the number of different pitches actually used in my recording of this movement posted on the North Coast audio server is 185, spread out over four octaves.

[2]Counterexample: “Zalgo’s Paradox.” [21]

8 About this document

The first version was posted August 2, 2015, to my personal Web site. This is the second version, substantially the same but with a few minor errors corrected and link URLs updated, posted October 1, 2017 in a Web log entry at https://northcoastsynthesis.com/blogs/news/notes-on-notes-on-the-plane. Subsequent to the first version I've left academic computer science to start a modular-synthesizer company. Your support of my business is what makes it possible for me to spend time on works like this one, so I hope you will buy many synthesizer modules, and mention that URL when you cite this document.

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