"HARMONIC LOGARITHMS" MAKE CENTS!

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Introduction: the "Non-cents" approach

Why do we use *cents* to measure music intervals? Professional instruments aim for a tuning accuracy of 0·06%, but with some people claiming they cannot hear a frequency change below 0·3%, these figures do not seem too awkward expressed as percentages.

A whole-tone is a frequency increase of about 12¼%. Six such consecutive steps constitute an octave:

$$(1 + {}^{12\cdot25}/_{100})^6 = 2\cdot0004$$

So, why do we need the complication of using *cents*?

Percentages are devious, non-linear devices used by politicians for statistics. If you increase a quantity by 10%, then subsequently reduce it by 10%, you do not end-up at the starting-point.

As Victorian prime-minister Disraeli succinctly put it:

"There are lies, damn lies and there are statistics."

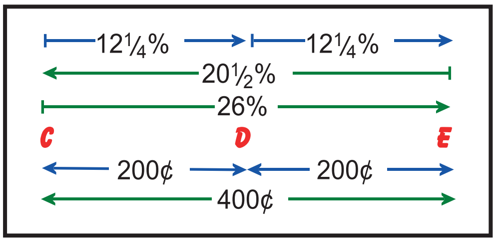
Because musical intervals are logarithmic, not linear, a major third (comprising two whole tones) is a 26% increase, not twice that of a whole tone:

$$(1 + {}^{26}/_{100})^3 = 2\cdot0004$$

Further, percentages are direction-sensitive: a major third is a frequency decrease of 20·5%:

$$(1 - {}^{20\cdot5}/_{100})^3 = 0\cdot5$$

(That figure, 0·5, should throw the spotlight on the problem in using percentages — an octave is a 100% increase but a 50% decrease!)



WHICH SYSTEM IS MORE CENTSIBLE?

*(Incidentally, symbols with strokes through them are reserved for currency — there is no official symbol for a musical cent. As cents are the "currency" of music, who gives a dime? P.S. To get the "¢" symbol on a keyboard without such a key, with **Num Lock** set, hold down **ALT** whilst typing **0162** on the numerical keypad. The **Lucida Sans Unicode** font also has the imposing symbol ₡, Unicode **20A1**).*

That's one small step ...

By definition, there are 100 cents in a semitone. We should be cautious in calling it 1% of a semitone, or in calling a quarter-tone half of a semitone (50 cents is more exactly the square-root of 100 cents), but when you're tone-deaf, it is of little consequence.

One cent is a ratio of (1: 1·0005778), a step increase of about 1 part in **1731**. One hundred consecutive steps constitute a semitone:

$$(1 + {}^{1}/_{1730\cdot7})^{100} = 1\cdot059464$$

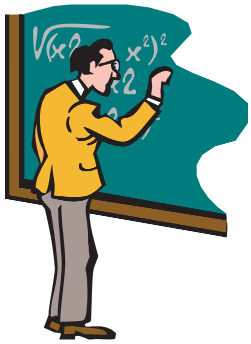
while 50 steps make a quarter-tone:

$$(1 + {}^{1}/_{1730\cdot7})^{50} = 1\cdot029303$$

(and just to check we're on the right track;

1·029303 × 1·029303 = 1·059464)

If you are suspicious of these claims, but cannot verify them on the back of an envelope, you can use your PC's calculator *(set to View/Scientific)* to see what 1200 of them amount to:



1731	*Enter "magic number"*
¹/x	*Take one part therein*
+ 1 =	*Form a step increase*
x^y 1200 =	*then iterate it.*
1.9998	*(result is close enough!)*

For those who like to show they are not tone-deaf, a more precise result of 2·000000078 is obtained using Magic Number = **1730·734**.

A "Magic Number" makes little cents

Although musical steps are logarithmic, if accuracy is not paramount, then the *Magic Number* can be used directly to convert percentages or frequency changes to/from **a few** cents...

1) *Within what frequency range must C = 523·25 Hz. be tuned to get within an accuracy of two cents?*

$$\frac{523\cdot25 \text{ Hz.}}{\mathbf{1731}} \times 2 \approx 0\cdot6 \text{ Hz.} \quad (\textit{viz. } \pm 0\cdot3 \text{ Hz.})$$

2) *How precise is my "tuning fork", which divides a 2·0 MHz. crystal oscillator by 4545 to produce 440·044 Hz.*

$$\frac{0\cdot044 \text{ Hz.}}{440 \text{ Hz.}} \times \mathbf{1731} \approx 0\cdot17 \text{ cents}$$

Don't get carried away with extravagant claims for your device, though: the tolerance on the oscillator is likely to be ten times greater!

3) *What is a frequency tolerance of ±0·5%, expressed in cents?*

$$0\cdot5\% \times \mathbf{17\cdot31} = \pm\ 8.6 \text{ cents}$$

i.e. $({}^{0\cdot5}/_{100} \times \mathbf{1731})$

4) *How does a crystal oscillator with a frequency tolerance given as 100 "ppm." convert to cents ?*

Parts per million (ppm) is just a way of expressing small percentages. Divide by 10,000 (i.e. back-space 4 decimal places) to get parts per *hundred*. Since 100 ppm is equal to 0.01 parts per hundred:

$$0.01\% \times \mathbf{17{\cdot}31} = 1{\cdot}7 \text{ cents}$$

5) *What is the difference in cents between a "Just Third" (5/4) and a "Tempered Third" ?*

$$\left(\frac{1{\cdot}25992 - 1{\cdot}25}{1{\cdot}25}\right) \times \mathbf{1731} \approx 13{\cdot}5¢$$

1731 replaces the multiplier "100" normally used in a percentage calculation. This method is not quite precise at these magnitudes, but (sadly) the bulk of the answer is true!

6) *What frequency is the quarter-tone above A=440 Hz. ?*

$$\left(\frac{440 \times 50}{1731}\right) + 440 \approx 452{\cdot}7 \text{ Hz.}$$

7) *Oh yeah, so what is a fifth (700¢) above 440 Hz.,then?*

$$\left(\frac{440 \times 700}{1731}\right) + 440 \approx 617{\cdot}9 \text{ Hz.}$$

Oops — even the tone-deaf might notice that! Should be about 660 Hz. (440 × 3/2). Magic number failure!

Stronger Magic needed

In film mythology, strong magic is invoked by using Latin. We invoke *logarithms*, especially *Natural* ones.

$$\textbf{Cents} = \text{Ln}(f_2/f_1) \times \mathbf{1731}$$

The frequency ratio of a tempered fifth is 1·4983, so testing this ratio in the above equation;

$$\text{Cents} = \text{Ln}(1{\cdot}4983) \times \mathbf{1731} = 0{\cdot}4043 \times 1731 = 699{\cdot}9¢$$

(Note: The "key" for the natural log. function on the PC's scientific calculator is marked ln, to the left of the MR (Memory Recall) button).

5) *What is the difference in cents between a "Just Third" (5/4) and a "Tempered Third" ?*

$$\text{Ln}\left(\frac{1{\cdot}25992}{1{\cdot}25}\right) \times \mathbf{1731} \approx 13{\cdot}7¢$$

This all seems plausible somehow, even if you do not quite understand it — and who understands magic? The formula seems to produce the right answers. In a scale system that we perceive as logarithmic, we take a frequency interval expressed as a ratio, and scale (i.e. multiply) the *logarithm* of that ratio by a number that is associated with a one-cent step.

One niggling doubt about this is why the formula only works with *natural*, not common (base 10), logs. Then again, music is *natural* and electro-music enthusiasts are anything but *common*!

Yet again, if you think hard about this and recall your school maths lessons, there is something odd going on here. We were taught that if you use logarithms in a calculation, then you have to take the *anti-log* of the result to get the true answer. How can we translate from one system into another by omitting this step?

The Appliance of Science

Enough of this mystic non-cents: it's time to consult a Scientific Nerd...

If an octave (ratio 2:1) comprises 1200 multiplicative steps, the general formula for any frequency ratio is:–

$$(f_2/f_1) = (2)^{¢/1200}$$

where ¢ is the interval expressed in cents.

Don't panic! All that frightening equation is saying is that for ¢ = 1200 cents, then the frequency ratio is 2.

When ¢ = 100, then the frequency ratio is $(2)^{1/12}$, the scientific way of writing the twelfth-root of 2.

Now, a really useful property of logarithms is that scary *exponents* (powers) are converted to less scary multiplication, thus:

$$\text{Log}\,(f_2/f_1) = ¢/1200 \times \text{Log}(2)$$

where f_2/f_1 is the frequency ratio, Log is the logarithm to **any** base, and ¢ is the interval expressed in cents.

$$\frac{1200}{\text{Log}(2)} \times \text{Log}\,(f_2/f_1) = ¢$$

For logarithms to base-10, $\text{Log}_{10}(2) = 0{\cdot}301$, so:

$$¢ = 3986{\cdot}314 \times \text{Log}_{10}(f_2/f_1)$$

For **natural** logarithms *(to the base of natural number* e *= 2·718282..)*, Ln(2) = 0·69315, and so

$$¢ = \frac{1200}{0{\cdot}69315} \times \text{Ln}\,(f_2/f_1)$$

$$¢ = \mathbf{1731{\cdot}234} \times \text{Ln}(f_2/f_1)$$

Doesn't that multiplier look a little familiar?!

It is not quite the same figure obtained on page 1, yet it is **not** a question of tolerances and approximations:

$$(1 + 1/1730{\cdot}734)^{1200} = 2{\cdot}0000000778$$

$$(1 + 1/1731{\cdot}234)^{1200} = 1{\cdot}9995998559$$

Testing both figures using a ratio of a semitone:

$$1730{\cdot}734 \times \text{Ln}(1{\cdot}0594631) = 99.97 \text{ cents}$$

$$1731{\cdot}234 \times \text{Ln}(1{\cdot}0594631) = 99.999997 \text{ cents}$$

Your author is not the scientific nerd depicted on the previous page, and does not know all the answers, but feel this is not a *discrepancy*, nor a co-incidence, but an aspect of the mathematical magic of infinite amounts of very small increases, best understood by mathematicians. (Any out there, reading this?)

Observe the similarity between the form:

$$(1 + {}^{1}/_{1730\cdot734})^{1200}$$

and a formula for obtaining the natural number, e :

$$(1 + {}^{1}/_{n})^{n}$$

where n is a mind-boggling huge number. In our case, 1 cent is not small enough nor 1731 large enough for the results to quite converge. But it's near enough for the tone-deaf!

Centitones, Millitones and Microtones

Luckily, the awkward term *'centitones'* is not used, as the term *"cents"* has already been defined to cover a semitone, not a whole-tone. In the proper scheme of things, *cents* would be called "*centi-semitones*". A "*centitone"* (if it existed) would therefore be equal to 2 *centi-semitones*, and 1 *centi-semitones* would equal a *semi-centitone*. That's far too confusing!

A ***"millitone"*** (mT) is instinctively a natural musical term — one thousandth of a whole-tone. There are 6 whole tones in an octave, i.e. 6000 millitones, and a semitone = 500mT; thus 1 cent = 5mT. With a quantization of 0·2 cents when using millitones, there should be no need to resort to decimal places.

One millitone is a step increase of about 1 part in **8656**. Five hundred steps constitute a semitone:

$$(1 + {}^{1}/_{8655\cdot67})^{500} = 1\cdot0594631$$

two thousand millitones make a major third:

$$(1 + {}^{1}/_{8655\cdot67})^{2000} = 1\cdot25992$$

3000 millitones make a tritone (= root 2):

$$(1 + {}^{1}/_{8655\cdot67})^{3000} = 1\cdot41421$$

etc. When using logarithms, simply multiply by an extra 5 to convert cents to millitones:

1) *What is the discrepancy between a "Just" fifth and a tempered fifth?*

(a) The "approximate" method for small intervals is :

$$\left(\frac{1\cdot5 - 1\cdot4983}{1\cdot5}\right) \times \mathbf{8656} \approx 10 \text{ mT}$$

(b) The "scientific" method is :

$$\text{Ln}\left(\frac{1\cdot5}{1\cdot49831}\right) \times \mathbf{8656.17} \approx 10 \text{ mT}$$

(i.e the result is approximately 2 cents). *Note that with the increased resolution (smaller steps), the two "magic number" multipliers converge — but only use the first method with fairly small intervals.*

2) *Why is the seventh harmonic said to be "out-of-tune"?*

The ratio of the 7th harmonic to the root note is 7:1

$$\text{Ln}(7) \times \mathbf{8656\cdot17} \approx 16{,}844 \text{ mT}$$

This is somewhere below 17 whole-tones: you need to then subtract multiples of 6 to appreciate that it lies somewhere short of the third octave (18000 mT = 18 tones). Even expressed as 3369 cents, you still have to subtract 2400 to reach the same conclusion.

By "folding-back" the seventh harmonic into the base octave (1·00 – 2·00), it forms a "Just seventh" with a relative pitch of $^{7}/_{4}$.

$$\text{Ln}(1\cdot75) \times \mathbf{8656\cdot17} \approx 4844 \text{ mT}$$

This is 4·844 whole-tones above the root, 1·156 tones below the octave. i.e. 156 mT below the equal-temperament seventh. *(But, of course, it is our scale that is out-of-tune, not the harmonic!)*

You could also compute it as

$$\text{Ln}(^{8}/_{7}) \times 1731\cdot234 = 231\cdot17 \text{ cents}$$

being the "distance" from the third octave (the 8 in the fraction) of the fundamental.

*The term **microtone** has already been usurped to mean "any interval less than a semitone". In Western music, this is principally the quarter-tone (250 millitones), so we can get the bizarre anomaly where a microtone =250 millitones!*

Harmonic Logarithms

As per the above example, a table of *Harmonic* (or *Music*) Logarithms (symbol: **L¢**) can be compiled for use in music theory by simply multiplying Natural logarithms by 1731·234 *(or by multiplying common logarithms by 3986·314)*. The table entry then gives the pitch of harmonics directly in cents, giving a clear indication of discords with the scale.

They can also be used to compare interval **ratios** against the equal-temperament scale. The *Just* scale has simple fractional intervals such as $^{9}/_{8}$, $^{10}/_{9}$, $^{5}/_{4}$, $^{4}/_{3}$, $^{5}/_{3}$, $^{15}/_{8}$. These are easily computed as, for example:

L¢($^{9}/_{8}$) = L¢(9) - L¢(8) = 3803.91 - 3600.00 = 203.9¢

The *Pythagorean* scale cascades eleven successive intervals of $^{3}/_{2}$, *(i.e $^{9}/_{4}$, $^{27}/_{8}$, $^{81}/_{16}$, etc.) viz*:

L¢($^{3}/_{2}$) = L¢(3) - L¢(2) = 1901.96 - 1200.00 = 701.96¢

(The Pythagorean scale is easily calibrated in cents by successively adding 701·96 and then subtracting 1200 whenever it exceeds that figure).

Similar tricks can be played with non-prime numbers: for example, the interval (55:49) can be calculated as:

L¢(5) + L¢(11) – L¢(7) – L¢(7) = 199.97¢

Ratio	L¢(R)	Note
1	0	KeyNote
1.0594631	100	Semitone
1.1224620	200	Tone
1.1892071	300	Min. 3rd.
1.2599210	400	Maj. 3rd.
1.3348399	500	Fourth
1.4142136	600	Tritone
1.4983071	700	Fifth
1.5874011	800	Min. 6th.
1.6817928	900	Maj. 6th.
1.7817974	1000	Min. 7th.
1.8877486	1100	Maj. 7th.
2.0000000	1200	Octave
2.1189262	1300	Semitone
2.2449241	1400	Tone
2.3784142	1500	Min. 3rd.
2.5198421	1600	Maj. 3rd.
2.6696797	1700	Fourth
2.8284271	1800	Tritone
2.9966142	1900	Fifth
3.1748021	2000	Min. 6th.
3.3635857	2100	Maj. 6th.
3.5635949	2200	Min. 7th.
3.7754973	2300	Maj. 7th.
4.0000000	2400	Octave
4.2378524	2500	Semitone
4.4898482	2600	Tone
4.7568285	2700	Min. 3rd.
5.0396842	2800	Maj. 3rd.
5.3393594	2900	Fourth
5.6568542	3000	Tritone
5.9932283	3100	Fifth
6.3496042	3200	Min. 6th.
6.7271713	3300	Maj. 6th.
7.1271897	3400	Min. 7th.
7.5509945	3500	Maj. 7th.
8.0000000	3600	Octave
8.4757048	3700	Semitone
8.9796964	3800	Tone
9.5136569	3900	Min. 3rd.
10.079368	4000	Maj. 3rd.
10.678719	4100	Fourth
11.313708	4200	Tritone
11.986457	4300	Fifth
12.699208	4400	Min. 6th.
13.454343	4500	Maj. 6th.
14.254379	4600	Min. 7th.
15.101989	4700	Maj. 7th.

Equal-Temperament Scale

h	L¢(h)	*Rel. to C_1*
1	0000.00	C_1
2	1200.00	C_2
3	1901.96	G_2 + 2¢
4	2400.00	C_3
5	2786.31	E_3 - 14¢
6	3101.96	G_3 + 2¢
7	3368.83	E_3 - 31¢
8	3600.00	C_4
9	3803.91	D_4 + 4¢
10	3986.31	E_4 - 14¢
11	4151.32	$F\sharp_4$ - 49¢
12	4301.95	G_4 + 2¢
13	4440.53	$A\flat_4$ + 41¢
14	4568.83	$B\flat_4$ - 31¢
15	4688.27	B_4 - 12¢
16	4800.00	C_5
17	4904.96	D_5 + 5¢
18	5003.91	D_5 + 4¢
19	5097.51	$E\flat_5$ - 2¢
20	5186.31	E_5 - 14¢
21	5270.78	F_5 - 29¢
22	5351.32	$F\sharp_5$ - 49¢
23	5428.27	$F\sharp_5$ + 28¢
24	5501.95	G_5 + 2¢
25	5572.63	$A\flat_5$ - 27¢
26	5640.53	$A\flat_5$ + 41¢
27	5705.86	A_5 + 6¢
28	5768.83	$B\flat_5$ - 31¢
29	5829.58	$B\flat_5$ + 30¢
30	5888.27	B_5 - 22¢
31	5945.04	B_5 + 45¢

Harmonic Logarithms

Non-prime logs that are not listed can be formed from factors ,e.g:
L¢(33) = L¢(11) + L¢(3)
L¢(34) = L¢(17) + L¢(2)
L¢(35) = L¢(7) + L¢(5)

Ratio		L¢(R)
89:84	1.0595238	100.10
107:101	1.0594059	99.91
196:185	1.0594595	99.99
55:49	1.1224490	199.98
44:37	1.1891892	299.97
63:50	1.26	400.11
99:70	1.4142857	600.09
127:80	1.5875	800.11
227:143	1.5874	800.01

Useful "8-bit" Ratios

	divisor	|D|	L¢(D/239)
C_9	239.00	239	0.00
B_8	253.21	253	98.55
$B\flat_8$	268.27	268	198.27
A_8	284.22	284	298.66
$A\flat_8$	301.12	301	399.30
G_8	319.03	319	499.86
$F\sharp_8$	338.00	338	600.02
F_8	358.10	358	699.54
E_8	379.39	379	**798.23**
$E\flat_8$	401.95	402	900.22
D_8	425.85	426	**1000.61**
$C\sharp_8$	451.17	451	1099.34

"TOGware" Logarithms

Max. Relative Error = 2.4¢

	divisor	|D|	L¢(D/349)
C_9	349.00	349	0.00
B_8	369.75	370	101.16
$B\flat_8$	391.74	392	201.16
A_8	415.03	415	**299.86**
$A\flat_8$	439.71	440	401.13
G_8	465.86	466	500.52
$F\sharp_8$	493.56	494	**601.54**
F_8	522.91	523	700.30
E_8	554.00	554	799.99
$E\flat_8$	586.95	587	900.16
D_8	621.85	622	1000.43
$C\sharp_8$	658.82	659	1100.46

"TOGware" Logarithms

Max. Rel. Error = 1.7¢ (0.1%)

	divisor	|D|	L¢(D/508)
C_9	508.00	508	0.00
B_8	538.21	538	99.33
$B\flat_8$	570.21	570	199.36
A_8	604.12	604	299.66
$A\flat_8$	640.04	640	399.89
G_8	678.10	678	499.75
$F\sharp_8$	718.42	718	**598.99**
F_8	761.14	761	699.68
E_8	806.40	806	799.14
$E\flat_8$	854.35	854	899.29
D_8	905.15	905	999.71
$C\sharp_8$	958.98	959	**1100.04**

"TOGware" Logarithms

Max. Rel. Error ≈ 1¢ (0.06%)